Current Psychology of Cognition/Cahiers de Psychologie Cognitive 2004, 22 (1), 41-82. Preprint

A Theory of the Graceful Complexification of Concepts and Their Learnability

Fabien Mathy* and Joël Bradmetz
Université de Reims ChampagneArdenne, France

Conceptual complexity is assessed by a multi-agent system which is tested experimentally. In this model, where each agent represents a working memory unit, concept learning is an inter-agent communication process that promotes the elaboration of common knowledge from distributed knowledge. Our hypothesis is that a concept's level of difficulty is determined by that of the multi-agent communication protocol. Three versions of the model, which differ according to how they compute entropy, are tested and compared to Feldman's model (Nature, 2000), where logical complexity (i.e., the maximal Boolean compression of the disjunctive normal form) is the best possible measure of conceptual complexity. All three models proved superior to Feldman's: the serial version is ahead by 5.5 points of variance in explaining adult inter-concept performance.

Computational complexity theories (Johnson, 1990; Lassaigne & Rougemont, 1996) provide a measure of complexity in terms of the computation load associated with a program's execution time. In this approach, called the structural approach, problems are grouped into classes on the basis of the machine time and space required by the algorithms used to solve them. A program is a function or a combination of functions. In view of developing psychological models, it can be likened to a concept, especially when y's domain $[y = f(x)]$ is confined to the values 0 and 1. A neighboring perspective (Delahaye, 1994) aimed at describing the complexity of objects (and not at solving problems) is useful for distinguishing between the"orderless, irregular, random, chaotic, random" complexity (this quantity is called algorithmic complexity, algorithmic randomness, algorithmic information content or Chaitin-Kolmogorov complexity; Chaitin, 1987; Kolmogorov, 1965) and the "organized, highly structured, information-rich" organized complexity (or Bennett's logical depth, 1986). Algorithmic complexity corresponds to the shortest program describing the object, whereas the logical depth of that program corresponds to its computation time. Working in this framework, Mathy and Bradmetz (1999) were able to account for learning and categorization in terms of computational complexity and logical depth. They devised a concept-complexity metric using a multi-agent system in which each agent represents one unit in working memory. In this model, a concept's dimensions are controlled by informationally encapsulated agents. Categorizing amounts to elaborating mutual knowledge in the multi-agent system. This model of complexity is grounded in an analysis of the complexity of the communicative process that agents must carry out to reach a state of mutual knowledge. Less concerned with psychological modelling, Feldman (2000) proposed a theory he presents as inaugural ("... the relation between Boolean complexity and human learning has never been comprehensively tested", p. 631), arguing that the complexity of a concept is best described by the compression of its disjunctive normal form (i.e., a disjunction of conjunctions of features). Feldman supported this idea with an experiment (after learning, recognize computer images of artificial amoebae of a given species that differ in size, shape, color, and number of nuclei), but no alterna-

*Correspondence concerning this article should be addressed to Fabien Mathy, Centre interdisciplinaire de recherche en linguistique et psychologie cognitive (CIRLEP), Université de Reims Champagne-Ardenne, UFR Lettres et Sciences Humaines, 57 rue Pierre Taittinger, 51096 Reims Cedex, France (e-mail: fabien.mathy@free.fr.

tive models were tested. Given the interest this model has aroused and its tempting simplicity, an experimental comparison with our model seemed indispensable.

Our model of working memory communications can be readily related to the working memory functions described in earlier research. When conceptual complexity is assessed using a multi-agent model, working memory can be jointly described by the number of units required and by the communication operations they must carry out. Working memory communications correspond to the operations controlled by the executive function; the number of units simply corresponds to the storage capacity. The processor, or executive function, has been described as a residual domain of ignorance (Baddeley, 1986, p. 225), although it has been widely studied in psychometrics for its assumed relationships with factor *g* (Crinella & Yu, 2000). The task-switching paradigm, for example, has been used to directly study executive control (Gilbert & Shallice, 2002). For instance, in Anderson's (1983) adaptive character of thought theory (ACT), the computational aspects of rule processing are opposed to declarative elements like facts and goals; see also Anderson's (1993) revised theory, ACT-R, and Newell's (1992) rule-production model, SOAR. However, in both models, no capacity limit is set for working memory. In Baddeley's (1976; 1986; 1990, 1992) well-known model, we find functional independence between the processor (limited capacity) and memory span (also limited), but most research based on this model has been confined to measuring memory span. Similarly, in his neo-Piagetian studies, Pascual-Leone (1970) reduced the executive component of working memory to the capacity for maintaining at least one unit in working memory. We shall see that the multi-agent models developed here incorporate both the executive component and the storage component.

The purpose of the present article is to provide an in-depth presentation of a theory of graceful complexification of concepts (i.e., continuous and non-saltatory/ non-stepwise; see McClelland, Rumelhart, & PDP Research Group, 1986) briefly addressed in Mathy & Bradmetz (1999), and to compare it to Feldman's (2000) theory. Like Feldman, we represent concepts as having a component-like structure obtained by concatenating a series of features (binary for the sake of simplicity; *n*-ary would simply lead to a combinatory burst). We also analyze Feldman's model and point out five of its shortcomings. Then we test and compare the two models experimentally.

Setting aside philosophical problems attached to a view of concepts that rests primarily on the composition of their underlying dimensions and the possibility of expressing them in a Boolean fashion (for a thorough discussion of this point, see Fodor, 1994), we retain the possibility of (i) using a set of examples defined by the number of dimensions (e.g., in three binary dimensions, shape (round or square), color (blue or red) and size (big or small), the space of examples contains 23 or 8 elements), (ii) using a Boolean expression to define subclasses of positive and negative examples that respectively do or do not represent the concept, and (iii) assigning each concept a function that separates the positive and negative examples (Figure 1). This view was initiated by

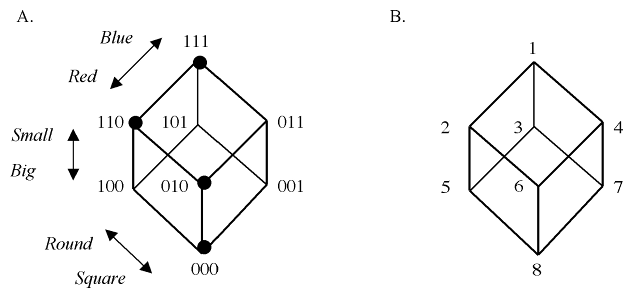


Figure 1. Illustration of a three-dimensional concept.

Bourne (1970); Bruner, Goodnow and Austin (1956); Levine (1966); Shepard, Hovland and Jenkins (1961).

The positive cases are shown as a black dot. Suppose the examples are geometrical objects. The upper and lower faces are *small* and *big* objects, the left and right faces are *round* and *square* objects, and the front and back faces are *red* and *blue* objects. Using classical formalism of propositional logic, we say that X is a positive example if:

$$X \equiv 111 \vee 110 \vee 010 \vee 000 \equiv (11*) \vee (0*0)$$

In other words, X is a positive example if X is $(Round \wedge Small) \vee (Square \wedge Red)$ (i.e., in natural language : (Round and Small) or (Square and Red)).

The information content of a sample of examples is then reduced to a set of pertinent information, in order to get an economic way of describing a concept. Learning in this case can be assimilated to a compression of information. The degree of this compression defines the compressibility of the concept. For example, the learner can reduce a concept to a rule. The multi-agent model proposed here has been developed to model the compression of a concept. We assume that each dimension of a concept is identified by a single agent. When communicating agents exchange minimal information to learn a concept, we measure the compression of the minimal communication protocol they used. This minimal communication protocol can correspond to the formalism of propositional logic with its use of disjunctive and conjunctive operators. The objective of the multi-agent model is to describe several ways to obtain a compressed propositional logic formula. Because the multi-agent system is an instantiation of a working memory model, these several ways to induce a concept will be developed according to different considerations of working memory. The model is presented in detail in appendices 1 until 4.

Parallel, Serial, and Random Versions of the Multi-Agent Model

Parallel version. The appendix 4 gives a detailed presentation of how multi-agent processing works. The cube-vertex numbering system used throughout this article is given in Figure 1B. In this presentation, it is assumed that entropy, i.e., the amount of information supplied (see Tables 7, 8, and 9), is calculated for each example presentation. This

Table 1
Cost of each example in Figure 1, for the random version.

	Examples								
Order	1	2	3	4	5	6	7	8	Total
FSC	2	2	2	3	2	3	3	3	20
FCS	3	3	3	2	3	2	2	2	20
SFC	2	2	2	3	2	3	3	3	20
SCF	3	2	2	3	3	2	2	3	20
CFS	3	3	3	2	3	2	2	2	20
CSF	3	2	2	3	3	2	2	3	20
Total	16	14	14	16	16	14	14	16	120

is a strong hypothesis that induces a parallel version of the model.

Serial version. One can also assume that entropy is calculated once and for all for a given concept, and that each example presented will be analyzed using the same communication protocol. In this condition, the order of the agents is identical for all examples of the concept. This is the way a traditional system of rules operates because communicating occurs in a serial manner and in a fixed order. To make the model's identification process clearer, each formula can be represented by a decision tree that gives the order of the speakers. Figure 2 presents these trees, which are read as follows: the root of the tree represents the choice to be made by the first speaker, Agent X, the branches down the next node are Y's, and the branches down the lowest node are Z's. A look at the simplest and most typical cases (1, 2, 6, 13) will make the others easier to understand.

Random version. In the third version of the model, called random, it is simply assumed that for each example presented, the agents express themselves in a random order[1]. Each concept can be assigned a number which is the mean number of agents that have to express themselves in order to identify the eight examples. As in the other versions, positive and negative examples are treated equally. In illustration, let us again consider the case in Figure 1 with the vertex numbers shown in Figure 1B. For each vertex, there are six possible speaking orders among the agents Form, Size and Color:

$$FSC, FCS, SFC, SCF, CFS, CSF.$$

For each of the eight vertices and the six orders, the identification process has a cost (cf., Table 1). The mean cost of identifying the eight examples of a given concept (i.e., the number of agents who will have to speak) is $120/8 = 15$, and the variance is 1. Figure 2 gives the costs of all concepts.

Communication Protocols

If we identify agents by their speaking turns X, Y, Z, and we retain the operators $\wedge$ (necessary) and [ ] (optional), every concept can be associated with a formula that corresponds to a communication protocol and expresses the total information-processing cost for its eight examples. Concepts 6, 9, and 12 in the parallel version of the model will be used here to illustrate how these formulas are determined. Each positive and negative example of Concept 6 possesses a homolog on one side, so two agents will always suffice to identify it. We therefore write $X^8 \wedge Y^8$ or simply $X \wedge Y$, where the absence of an exponent is equal to an exponent of "8" by default. This formula means that for the eight cases of the concept, two agents will be necessary and sufficient each time. For Concept 9, we can see that all negative examples have a homolog on one side so they can be identified by two agents, but the positive examples are isolated so they will need three agents. This is written $X \wedge Y[Z]^2$, meaning that two agents will always be necessary, and in two cases they will not be sufficient and a third agent will have to contribute. Looking at Concept 12, we find that four cases (all negative) are located on a side with a homolog so two agents will be necessary and sufficient, but the other four are isolated and therefore require three agents. This is denoted $X \wedge Y[Z]^4$. The formulas for the serial and parallel versions are given in Figure 2.

Feldman's Model (2000)

Feldman examines positive examples only, which he enumerates in disjunctive normal form (DNF) (see also Feldman, 2003, for a complete classification of Boolean concepts from one to four dimensions). Illustrating with the concept in Figure 1 again, if we distinguish $f(\mathit{round})$, $f'(\mathit{square})$, $s(\mathit{big})$, $s'(\mathit{small})$, $c(\mathit{red})$, and $c'(\mathit{blue})$, the positive examples are written

$$1 = fs'c',\ 2 = fs'c,\ 6 = f's'c,\ 8 = f'sc$$

and the concept is written

$$(f \wedge s' \wedge c') \vee (f \wedge s' \wedge c) \vee (f' \wedge s' \wedge c) \vee (f' \wedge s \wedge c)$$

Using a heuristic that is not described in depth, Feldman states that the maximal compression of this DNF would be $c \wedge (s \wedge f)' \vee (c' \wedge s' \wedge f)$. Here, the notation system of propositional logic, based on Morgan's law, is used to go from one connective to the other by means of negation (denoted by an apostrophe): $(s \wedge f)' \equiv (s' \vee f')$. Then the author counts the number of letters in the compressed formula and draws from it the complexity index for that concept. Apart from its predictive value, this model calls for five remarks.

1. It moves immediately from the DNF of a concept to its complexity, without trying to support the psychological plausibility of this transition, that is, without attempting to come closer to a plausible way of functioning for working memory, even a relatively global and abstract one.

2. For the model to be granted the universality it claims to have, certain choices made in elaborating the formulas need to be developed and supported. Why was the DNF chosen? What makes the connectives $\wedge$ and $\vee$ superior to others for modelling categorization? What compression heuristics are used? Why is an arbitrary, fixed order employed for enumerating the binary features of the concept?

[1] This principle is similar in some ways to the one defended by Pascual-Leone (1970), who uses the Bose-Einstein occupancy model of combinatorial analysis to describe the activation of working-memory units.

3. There is an error in the model, related both to the opacity of the compression technique and to the arbitrary order adopted for concept features. A simple method called Karnaugh-Veitch diagrams, which can be easily applied by hand (Harris & Stocker, 1998; Vélu, 1999), indicates that Concept 7 (precisely the one used as an example) has a maximal compression of 4, not 6 as the author states, since the formula $(f \wedge s' \wedge c') \vee (f \wedge s' \wedge c) \vee (f' \wedge s' \wedge c) \vee (f' \wedge s \wedge c)$ can be reduced to $(s' \wedge f) \vee (c \wedge f')$, which describes the *small round* examples and the *red square* examples.

4. The author briefly mentions prototype theories in his conclusion, without indicating how his construction could be a model of them.

5. It is not clear why positive and negative examples are not treated equally and why they even constitute a complexity factor independently of Boolean complexity. This point is simply linked to the method used by the author and is not an intrinsic characteristic of concepts. In our method, as we shall see, symmetrical processing of positive and negative cases eliminates this source of variation.

The concept-complexity indexes derived from Feldman's model (corrected for Concept 7) are given in Figure 2. To facilitate comparison with the other indexes, we multiplied them by 8.

Concept Learnability

A concept is a discriminant function in a space of examples. The simplest discrimination is linear separability, the kind a perceptron can achieve without a hidden layer. Linear separability supplies a measure of complexity, but it is insufficient because it performs an undifferentiated classification that puts a whole range of cases with interactions of variable complexity under a single label, *inseparable*. The multi-agent model conveniently offers the possibility of assigning each concept a particular Boolean function, which, provided one is able to order those functions – we shall see later how a lattice can so just that – considerably enriches the separable versus non-separable model. In this case, every function used has a corresponding Vapnik-Chervonenkis dimension (VC) (see Boucheron, 1992) and exhibits better case discrimination[2].

Between learning and knowledge activation there is identity of form. When a learning master supplies answers, the subject learns to identify the stimulus and to associate it with a subclass (positive or negative); in other words, novices do the same thing as when they know the concept and activate their own knowledge. One can thus assume logically that the time taken to identify an example and assign it to the correct subclass once the concept is learned is proportional to the concept's complexity, and also that the learning time, i.e., the time taken to memorize the subclass to which each example belongs, is proportional to the concept's complexity as well.

A hierarchy of concept complexity up to three dimensions can be proposed, based on the ordering of multi-agent formulas in a Galois lattice (Appendix 3, Figure 12). However, it is more practical to assign an overall complexity index to a concept by taking the sum of all calls to all agents that occur while the concept's eight examples are being identified. These indexes, which indicate the logical depth of the concept, are given in Figure 2 for the various versions of the model.

Now let us present the method that enables us to compare the four measures of conceptual complexity: multi-agent serial, parallel, and random complexity, and Feldman's Boolean complexity.

METHOD

Subjects

Seventy-three undergraduate and graduate students (29 men and 44 women) between the ages of 18 and 29 (mean age: 21 years 7 months) participated in the experiment.

Procedure

A computer-assisted learning program was written (available at http://fabien.mathy.free.fr/). A standard concept-learning protocol was used: examples were presented in succession to the subject, who had to sort them by putting the positive examples in a "briefcase" and the negative examples in a "trash can". Feedback was given each time. The examples were generated from three dimensions: (i) shape (square, oval, or cross), (ii) color (red, blue, purple, or green), and (iii) the type of frame around the colored shape (diamond or circle). Many shapes and colors were used so that on each new concept, the examples would look different enough to avoid interference and confusion with the preceding concepts. For each concept, the dimensions were of course used only in a binary way in the entire set of examples (e.g., square vs. oval or square vs. cross).

A set of examples was composed of a subset of positive examples (*ex*+); its complementary subset was composed of negative examples (*ex*−). This partition defined the target concept. When the image displayed on the computer depicted an *ex*+, the learner had to click on the briefcase drawn in a window designed for that purpose; an *ex*− had to be put in the trash can shown in another window. Clicking the mouse in either of these windows caused the other window to disappear so that the feedback would be clearly associated with the clicked window. Each correct answer was rewarded with a "Bravo" heard and displayed in the feedback window. When an incorrect answer was given, a message was displayed saying "Not in the briefcase" or "Not in the trash can", depending on the case. All feedback messages remained on the screen for two seconds. A "Too late" message was displayed after eight seconds if the subject still had not

[2] Staying within the domain of linear separability as it is computed by a perceptron without a hidden layer, the VC dimension, for 3 dimensions, is equal to 4, that is, the perceptron can separate all subsets of 4 vertices obtained by bi-partitioning the vertices of a cube, provided the 4 vertices indeed define 3 dimensions and not just the 4 corners of a square. For cube example sets with more than 4 members, linear separability still may be possible, depending on the case, but it is not necessarily so.

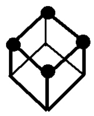
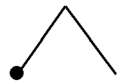
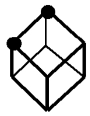
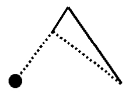
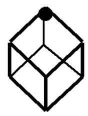
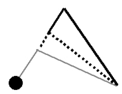
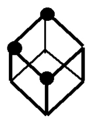
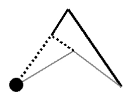
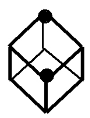
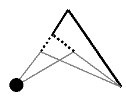
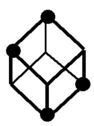
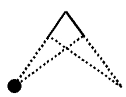
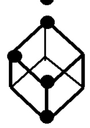
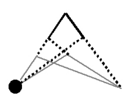
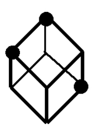
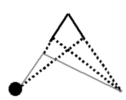
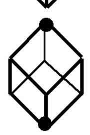
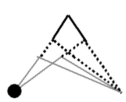
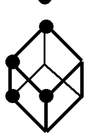
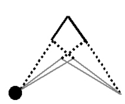
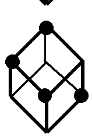
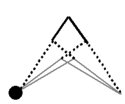
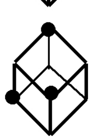
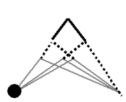
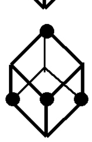
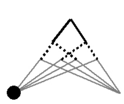
N°.	Concept	P	S	Tree	PC	SC	RC	FC
1		X^8	X^8		8	8	12	8
2		$X^8\ [Y]^2$	$X^8\ [Y]^4$		10	12	12	16
3		$X^8\ [Y[Z]^1]^1$	$X^8\ [Y[Z]^2]^4$		10	14	10.5	24
4		$X^8\ [Y[Z]^{1/3}]^4$	$X^8\ [Y[Z]^2]^4$		12.3	14	13.5	24
5		$X^8\ [Y[Z]^2]^4$	$X^8\ [Y \wedge Z]^4$		14	16	14	40
6		$X^8 \wedge Y^8$	$X^8 \wedge Y^8$		16	16	16	32
7		$X^8 \wedge Y^8$	$X^8 \wedge Y^8\ [Z]^4$		16	20	15	32
8		$X^8 \wedge Y^8\ [Z]^1$	$X^8 \wedge Y^8\ [Z]^2$		17	18	15.5	40
9		$X^8 \wedge Y^8\ [Z]^2$	$X^8 \wedge Y^8\ [Z]^4$		18	20	15	48
10		$X^8 \wedge Y^8\ [Z]^2$	$X^8 \wedge Y^8\ [Z]^4$		18	20	15	48
11		$X^8 \wedge Y^8\ [Z]^2$	$X^8 \wedge Y^8\ [Z]^4$		18	20	16	48
12		$X^8 \wedge Y^8\ [Z]^4$	$X^8 \wedge Y^8\ [Z]^6$		20	22	16.5	64
13		$X^8 \wedge Y^8 \wedge Z^8$	$X^8 \wedge Y^8 \wedge Z^8$		24	24	18	80

Figure 2. Formulas and decision trees for concepts up to three dimensions. *Note. P*: communication protocol formula for the parallel model. *S*: communication protocol formula for the serial model. *PC*, *SC*, *RC*, *FC*: complexity index for the parallel, serial, random, and Feldman (2000) models, respectively. (*X*'s choices are shown as a solid line, *Y*'s as a black dotted line, and *Z*'s as a grey dotted line).

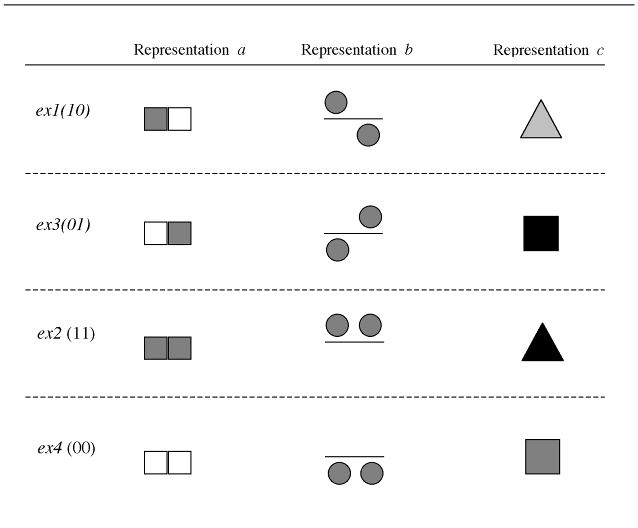


Figure 3. Possible ways of representing Boolean dimensions as physical dimensions. *Note.* Representation *a*: numerical facilitation. Representation *b*: spatial and numerical facilitation. Representation *c*: separable dimensions are amalgamated.

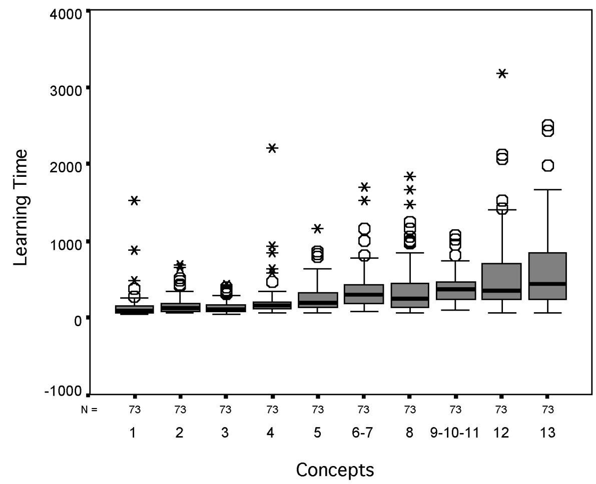


Figure 4. Boxplots of learning times for the 13 concepts.

clicked on the briefcase or trash can, and the next example was displayed; this message kept the game going and avoided unnecessary use of time. The time limit of eight seconds was assessed by means of various surveys which yielded a late response rate of less than 1%. Eight seconds is obviously not an absolute response time but simply served here as a means of putting all subjects in the same situation. The concept-learning criterion was deemed to be reached when the subject correctly sorted 16 consecutive examples. Every time a correct answer was given, one of 16 small squares in the progress bar was filled in (in black). Subjects could therefore see their progress at any time. Three black squares were erased if the subject responded too late. A mistake erased all previously filled-in squares and reset the learning counter at zero.

All three dimensions were represented in a single figure (e.g., an example could be a *red square* with a *diamond* around it) in order to avoid spatial representations (in case of an example represented by three present or absent forms in a row). Spatial representations would have facilitated learning (see examples in Figure 3). The dimensions were also representative of separable dimensions permitting independent dimension processing (as opposed to integral dimensions; see Garner, 1974).

The dependent variables were (i) the time taken to learn the concept (T), (ii) the total number of responses (i.e., the number of examples used) (R), and (iii) the number of errors made (E).

RESULTS

Table 2 shows that the correlations between the three dependent variables were greater than .95, $p < .01$. This finding allowed us to look solely at the total learning time, which was the most highly correlated with the complexity indexes for the four models. To avoid any effects brought about by the extreme scores (which can easily be seen in Figure 4), the learning times for each subject were transformed into ranks (TR) by attributing ranks 1 to 13 to the concepts (rank 13 was given to the concept learned the fastest so the coefficients in the regression analyses would remain positive).

Now, to compare the four models and identify the sources of variation in performance, we chose a linear regression analysis. Two explanatory variables (independent by construction) were selected: the presentation rank (*PR*) of a concept (drawn at random during the experiment) and the complexity index of each model (*SC*, *PC*, *RC*, and *FC*, for the complexity indexes of the serial, parallel, random, and Feldman models, respectively). The presentation rank (*PR*) was included because learning time decreased as the rank for learning a concept increased ($F(12,72) = 9.7, p < .001$), as indicated in Figure 5. The dependent variable was the total learning time rank (TR) of the concept among the 13 that each subject learned. The regression analysis was applied to the four complexity indexes. The results are presented in Figure 6. The amounts of variance explained (R^2) by computing a multiple regression on the concept-presentation ranks (*PR*) and on each complexity index were .423, .418, .400, and .363 for indexes *SC*, *PC*, *RC*, and *FC*, respectively. All of the analyses of variance on the multiple correlations were significant ($F(2,946) = 347, 340, 315,$ and 270, respectively, $p < .001$). The results of the *t*-tests applied to paired correlations (i.e., to the regression line coefficients) between each of the complexity indexes and the learning time ranks (TR) are given in Table 3. The models were ordered as follows: $S > P > R > F$. Thus, all three multi-agent models turned out to be superior to Feldman's model (2000). The serial, parallel and random models (which were statistically indistinguishable) are significantly better than Feldman's model.

Table 2
Correlations between the complexity indexes and the scores.

	SC	*PC*	*RC*	*FC*	*TR*	*T*	*R*
PC	.958**						
RC	.841**	.940**					
FC	.925**	.957**	.843**				
TR	.576**	.574**	.559**	.533**			
T	.363**	.380**	.374**	.364**	.646**		
R	.351**	.367**	.359**	.353**	.637**	.978**	
E	.335**	.358**	.357**	.339**	.612**	.953**	.965**

Note. *TR*: total learning time rank. *T*: total time. *R*: number of responses. *E*: number of errors. *SC*: serial complexity. *PC*: parallel complexity. *RC*: random complexity. *FC*: Feldman's complexity (2000).**: correlation is significant at the 0.01 level (two-tailed).

Table 3
Values of Student's t between correlations.

	$r_{\text{TR.PC}}$	$r_{\text{TR.RC}}$	$r_{\text{TR.FC}}$
$r_{\text{TR.SC}}$	0.26	1.15	4.16**
$r_{\text{TR.PC}}$		1.63	5.26**
$r_{\text{TR.RC}}$			1.73

Note. *: significant at $p < .05$. **: significant at $p < .01$. *TR*: learning-time rank. *SC*: serial complexity. *PC*: parallel complexity. *RC*: random complexity. *FC*: Feldman's complexity (2000). These t-values on non-independent samples were calculated using Steiger's (1980) formula (see Howell, 1997, p. 300).

The superiority of the serial model points out the merits of modelling information processing in working memory using models that process information in a fixed order.

Regressions on the mean learning times were also calculated. Let us compare the two models that differed the most as to their prediction of learning time (Feldman's model and the serial model). The amount of variance explained by Feldman's model in this condition was greater ($R^2 = .815,\ F(1,11) = 49,\ p < .001$) than the serial model ($R^2 = .808,\ F(1,11) = 46,\ p < .001$), although the difference between the regression line coefficients was nonsignificant ($t(11) = -0.8,\ NS$). In illustration, Figure 7 shows the regression line between the mean learning times and the serial complexity indexes of the concepts. This finding indicates that, despite the greater readability of the results, regression calculations on mean learning times (the basis of Feldman's calculations, 2000) do not point out which models best fit the data[3].

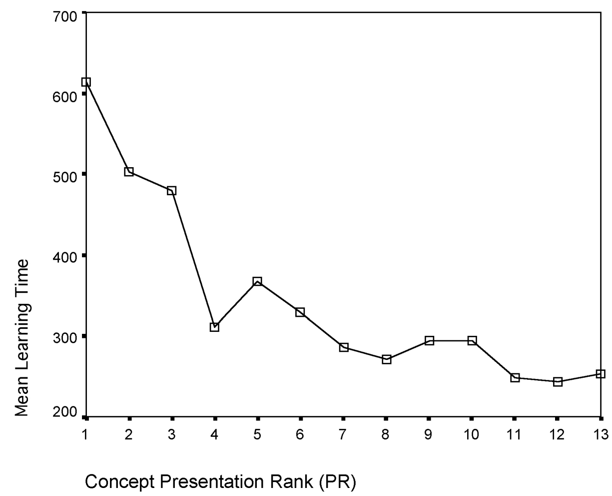


Figure 5. Mean learning time in seconds, by presentation rank of the 13 concepts.

DISCUSSION

One of the goals of this study was to evaluate Feldman's (2000) model of conceptual complexity with respect to a series of multi-agent models developed to be analogous to the functioning of working memory. The results showed that all three multi-agent models tested are superior to Feldman's (2000) for the regression of learning-time over the set of three-dimensional concepts. The difference between Feldman's model and the multi-agent models lies not only in the fact that the latter take the complexity of a concept into account in terms of positive and negative examples (unlike Feldman's model which looks solely at positive examples), but also in their clarification of dimension processing.

The inherent advantage of multi-agent models is that they allow one to address the question of the nature of information processing (serial, parallel, or random). Our results showed that the serial model is the best model because it imposes a fixed information-processing order. One reason why it prevails over the other three is certainly due to relatively constant patterns within noun phrases in natural languages (e.g., *big red round* instead of *round red big*). The phrase's stability seems to be rooted in stylistic considerations, which impose a certain order when features are being enumerated. This would fix their order during stimulus rehearsal. Besides, in the three multi-agent models developed here, the com-

[3] Pascual-Leone (1970) obtained excellent fits between theoretical and experimental curves using this same technique. By taking interindividual variability into account, Bradmetz and Floccia (submitted) showed with a LISREL model that a large part of the variance is not explained by Pascual-Leone's model and that the data fit should therefore be rejected.

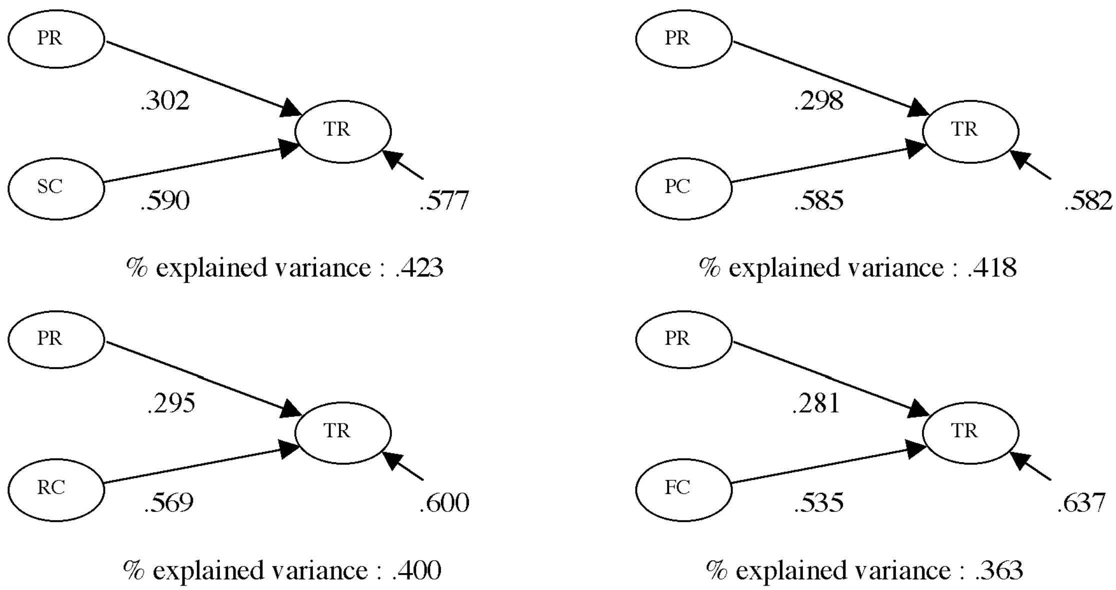


Figure 6. Linear regression for the four complexity models. *Note. PR*: presentation rank. *TR*: learning-time rank. *SC*: serial complexity. *PC*: parallel complexity. *RC*: random complexity. *FC*: Feldman's complexity (2000).

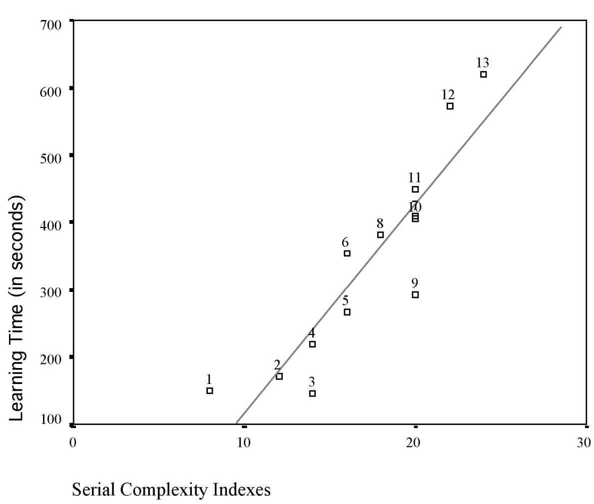


Figure 7. Relationship between mean learning time and serial concept-complexity indexes.

munications corresponding to the executive part of working memory processing is represented by decision trees. Because it imposes an unchangeable order for nodes at different tree depths, the serial multi-agent system can be reduced to a production system like that found in the symbolic approach to reasoning (Newell, 1983, 1990, 1992; Newell & Simon, 1972; Anderson, 1983; Holland, Holyoak, Nisbett, & Thagard, 1986). In the latter approach, reasoning is based on rules whose complexity depends on the nested hierarchization of a set of variables (an idea also found in developmental psychology in Zelazo, Frye, & Rapus, 1996). Our parallel and random multi-agent models, however, go beyond the traditional representation in decision-tree format, which imposes a predefined order on the dimension hierarchy. As for the random model, it has even fewer order constraints than the parallel model: it does not need to compute entropy since agents randomly put in their information until the examples are identified. In this case, the amount of information needed only depends a posteriori on a concept's entropy (since the savings in terms of fewer speaking turns still depends on the concept's structure) and not on an a priori calculation of the concept's entropy. Feldman's model makes a clear distinction between the number of pieces of information (connected by conjunctions) needed to classify examples in disjunctive format (1, 2, or 3 pieces). However, in addition to information quantity, multi-agent models distinguish several operating modes by introducing the idea of information ordering. To draw an analogy, multi-agent models would not only account for the number of digits (or chunks) to memorize in a telephone number, but also problems of order and simultaneity in chunk addressing or retrieval. This kind of information is critical because (depending on the chosen parameters) it may lower the amount of information that has to be supplied to classify an example. The amount of information required per example is therefore not an absolute measure derived from disjunctive normal forms, but a relative measure that depends upon the type of processing carried out on the information.

Another point concerns the originality of representing working memory processing in terms of communication protocol formulas. The expressive power of communication formulas for describing Boolean functions is no greater than any other type of formalization, but their compressibility is greater. By reducing the decision-tree structure to binary communication operators, formulas offer a compressed representation of working memory operations. The agents who give the greatest amount of information are chosen first. One advantage of this approach is that it brings out the correspondence between the entropy measure and a classification of simple communication situations (choice, simple interactions, complete interactions). A communication formula represents the necessary dimensions only once in its written form. This formal simplicity stems from the fact that each communication protocol corresponds to a speaking turn calculated by a measure of information gain (based on an entropy calculation). A formalization like this (in terms of communication formulas) is simpler to read than the disjunctive normal forms proposed by Feldman. For instance, our multi-agent system models Concept 13 as $X \wedge Y \wedge Z$, whereas Feldman's formula is $x(y'z \vee yz') \vee x'(y'z' \vee yz)$. We also find a correspondence between the number n of $\wedge$ operators and the interaction structures of order n that exist in situations with $n+1$ dimensions (e.g. the formula $X \wedge Y \wedge Z$ describing a second-order interaction).

This study still has some limitations when it comes to evaluating the parallel, serial, and random models. It is difficult to validate one of the three models with the inter-concept comparison method developed here. Mathy (2002) proposed conducting a study grounded on an intra-concept comparison method that would be better able to distinguish the models. Indeed, the way multi-agent models function is such that there is a different number of agents for each example of a concept. By reading the communication protocol formulas (or tables for the random model), it suffices to establish, for each example of a concept, the number of agents needed to classify it. An example that requires more agents in order to be categorized (i.e., one representing a longer path in the decision tree) will correspond to higher response times in an application phase of an already-learned concept. One would no longer measure the tree-construction time but the time needed to navigate in an induced tree.

RÉSUMÉ

Cet article propose un modèle et une évaluation expérimentale de la complexité des concepts au moyen d'un système multi-agent dans lequel chaque agent représente une unité en mémoire de travail. Ce modèle conçoit l'apprentissage de concepts comme une activité de communication inter-agents permettant de passer d'une connaissance distribuée explicite à une connaissance commune. L'hypothèse est que le degré de difficulté d'une tâche de conceptualisation est déterminé par celui du protocole de communication inter-agents. Trois versions du modèle, différant selon le mode de calcul de l'entropie du système, sont testées et sont comparées au modèle que Feldman (Nature, 2000) présente comme définitif, en réduisant la complexité d'un concept à la compression maximale de sa forme disjonctive normale booléenne. Les trois versions du modèle se révèlent supérieures au modèle de Feldman : la version séquentielle gagne 5,5 points de variance dans l'explication des performances inter-concepts de sujets adultes.

REFERENCES

Anderson, J.R. (1983). *The architecture of cognition*. Cambridge MA, Harvard University Press.

Apostel, L. (1963). Structure et genèse. In L. Apostel, J-B Grize, S. Papert & J. Piaget. *La filiation des structures*. Paris: Presses Universitaires de France.

Axelrod, R. (1997). *The complexity of cooperation: Agent-based models of competition and collaboration*. Princeton, NJ: Princeton University Press.

Baddeley, A. (1976). *The psychology of memory*. New York, NY: Basic Books.

Baddeley, A. (1986). *Working memory*. Oxford: Clarendon Press.

Baddeley, A. (1990). *Human memory: Theory and practice*. Boston: Allyn and Bacon.

Baddeley, A. (1992). Working memory. *Science*, *255*, 556-559. (French translation: *La mémoire Humaine*. Grenoble: Presse Universitaire de Grenoble, 1993).

Bennett, C. H. (1986). On the nature and origin of complexity in discrete, homogeneous, locally-interacting systems. *Foundations of physics, vol. 16*, No. 6, 585-592.

Boucheron, S. (1992). *Théorie de l'apprentissage*. Paris: Hermès.

Bourne, L.E. Jr. (1970). Knowing and using concepts. *Psychological Review, 77*, 546-556.

Bradmetz, J. & Floccia, C. (submitted). *Comparison of four information processing models in Pascual-Leone's CSVI task.*

Brazier, F., Dunin-Keplicz, B., Jennings, N.R. & Treur, J. (1995). Formal Specification of multi-agent systems: a real-world case. *Proceedings of the First International Conference of Multi-Agent Systems* (pp. 25-32). San Francisco, CA: MIT Press.

Bruner, J.S., Goodnow, J.J., & Austin, G.A. (1956). *A study of thinking*. New York, NY: Wiley.

Bryant, R.E. (1986). Graph-based algorithms for Boolean function manipulation. *IEEE Transactions on Compilers*, *C-35*(8).

Burmeister, B. & Sundermeyer, K. (1990). COSY: towards a methodology of Multi-Agent Systems. *International Working Conference of Cooperating Knowledge-Based Systems*, Keele, UK.

Chaitin, G.J. (1987). *Information, randomness and incompleteness: papers on algorithmic information theory*. New York, NY: World Scientific.

Cowan, N. (2001). The magical number 4 in short-term memory: A reconsideration of mental storage capacity. *Behavioral and Brain Science, 24*, 1, 87-185.

Crabtree, B., & Jennings, N.R. (Eds). (1996). *Proceedings of the 1st International Conference on Practical Applications of Intelligent Agents and Multi-Agent Technology.*

Crinella, F.M., & Yu, J. (2000). Brain mechanisms and intelligence. Psychometric g and executive function. *Intelligence, 27*, 4, 299-327.

Davey, B.A. & Priestley, H.A. (1990). *Introduction to lattices and order*. Cambridge MA: Cambridge University Press.

Delahaye, J. P. (1994). *Information, complexité et hasard*. Paris: Hermès.

Epstein, J.M., & Axtell, R. (1996). *Growing artificial societies*. Cambridge MA: MIT Press.

Fagin, R., Halpern, J.Y., Moses, Y., & Vardi, M.Y. (1995). *Reasoning about knowledge*. Cambridge, MA: MIT Press.

Feldman, J. (2000). Minimization of Boolean complexity in human concept learning. *Nature, 407*, 630-633.

Feldman, J. (2003). A catalog of Boolean concepts. *Journal of Mathematical Psychology, 47*, 75-89.

Ferber, J., & Gutknecht, O. (1998). A meta-model for analysis and design of organizations in Multi-Agent Systems. *Third International Conference on Multi-Agent Systems*, ICMAS98, pp. 128-135, Paris, France.

Ferber, J. (1999). *Multi-Agent Systems, an introduction to distributed artificial intelligence*. Harlow: Addison-Wesley.

Fischer, K.W. (1980). A theory of cognitive development: the control and construction of hierarchies of skills. *Psychological Review, 87*, 477-531.

Fodor, J. (1994). Concepts: a potboiler. *Cognition, 50*, 95-113.

Ganter, B. & Wille, R. (1991). Applied lattice theory: formal concept analysis, in G. Grätzer (Ed), *General lattice theory*, Basel: Birkhaüser.

Garner, W. R. (1974). *The processing of information and structure*. Potomac, MD: Lawrence Erlbaum.

Gell-Mann, M. (1994). *The quark and the jaguar: Adventures in the simple and the complex*. New York, NY: Freeman and Company.

Gilbert, N., & Conte, R. (1995). *Artificial Societies*. London: UCL Press.

Gilbert, S.J. & Shallice, T. (2002). Task switching: a PDP model. *Cognitive Psychology, 44*, 297-337.

Halford, G.S., Wilson, W.H., & Phillips, S. (1998). Processing capacity defined by relational complexity: implications for comparative, developmental, and cognitive psychology. *Behavioral and Brain Sciences, 21*, 803-864.

Harris, J. W. & Stocker, H. (1998). *Handbook of mathematics and computational science*. New York, NY: Springer-Verlag.

Holland, J.H., Holyoak, K.J., Nisbett, R.E. & Thagard, P.R. (1986). *Induction: processes of inference, learning and discovery*. Cambridge, MA: MIT Press.

Howell, D.C. (1997). *Statistical methods for psychology* (4th edition). New York: Duxbury. (French translation by M. Rogier, *Méthodes statistiques en sciences humaines*, Bruxelles, De Boeck, 1998).

Johnson, D. (1990). A catalog of complexity classes, in Van Leeuven (Ed), *Handbook of theorical computer science* (pp. 67-161). Amsterdam: Elsevier Science Publishers, 67-161.

Kendall, E.A., Malkoum, M.T., & Jiang, C.H. (1995). A methodology for developing Agent Based Systems, in C. Zhang & D. Lukose (Eds), *First Australian Workshop on Distributed Artificial Intelligence*.

Kolmogorov, A. N. (1965). Three approaches for defining the concept of information quantity. *Information Transmission, vol 1*, pp 3-11.

Lassaigne, R., & De Rougemont, M. (1996). *Logique et complexité*. Paris: Hermès.

Levine, M. (1966). Hypothesis behavior by humans during discrimination learning. *Journal of Experimental Psychology, 71*, 331-338.

Mathy, F. & Bradmetz, J. (1999). Un modèle multiagent de la complexité conceptuelle. In M.P. Gleizes & P. Marcenac (Eds.), *Ingénierie des systèmes multi-agents: acte des 7e Journées Francophones d'Intelligence Artificielle et Systèmes Multi-Agents*. Paris: Hermès. (pp. 343-345).

Mathy, F. (2000). *L'apprenabilité des concepts évaluée au moyen d'un modèle multi-agent de la complexité des communications en mémoire de travail*. Unpublished doctoral dissertation, Université de Reims, France.

McClelland, J.L. & Rumelhart, D.E. & PDP Research Group. (1986). *Parallel distributed processing* (2 vol). Cambridge : MIT Press.

Minsky, M. L. (1985). *The society of mind*. New York, NY: Simon and Schuster (French translation: *La société de l'esprit*. Paris: InterEditions, 1988)

Müller, J.P. (1996). *The design of intelligent agents*. Lectures notes in artificial intelligence, Volume 1177. New York, NY: Springer-Verlag.

Newell, A., & Simon, H.A. (1972). *Human Problem Solving*. Englewood Cliffs, NJ: Prentice Hall.

Newell, A. (1983). *Production systems: Models of control structures*. In W.G. Chase, (Ed), Visual information processing. New York, NY: Academic Press.

Newell, A. (1990). *Unified theories of cognition*. Cambridge, MA: Harvard University Press.

Newell, A. (1992). SOAR as an unified theory of cognition: issues and explanations. *Behavioral and Brain Science, 15*, 464-492.

Nozaki, A. & Anno, M. (1985). *Anno's hat tricks*. New York: Philomel Books. (Tr. Fr., *Le jeu des chapeaux*. Paris: Flammarion, 1991).

Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piagets developmental stages. *Acta Psychologica, 32*, 301-345.

Quinlan, J.R. (1986). Induction of decision trees. *Machine learning, 1*, 81-106.

Ruth, M., & Ryan, M. (2000). *Logic in computer science*. Cambridge, UK: University Press.

Selfridge, O. (1959). Pandemonium: a paradigm for learning. In *Symposium on the Mechanization of Thought Processes*. London: HM Stationery Office.

Shepard, R. N., Hovland, C. L., & Jenkins, H. M. (1961). Learning and memorization of classifications. *Psychological Monographs, 75*, n13, 1-42.

Steiger, J.H. (1980). Tests for comparing elements of a correlation matrix. *Psychological Bulletin, 87*, 245-251.

Vélu, J. (1999). *Méthodes mathématiques pour l'informatique*. Paris: Dunod.

Zelazo, P. D., Frye, D. & Rapus, T. (1996). An aged-related dissociation between knowing rules and using them. *Cognitive Development, 11*, 37-63.

Received June 11, 2003 Accepted November 24, 2003

APPENDICES

Appendix 1. Multi-Agent Models

Multi-agent models are collective problem-solving methods. Although this idea is relatively old in psychology (Minsky, 1985; Selfridge, 1959), it was not until recently that simulations of agent societies and their evolution in computer science were developed (Brazier, Dunin-Keplicz, Jennings, & Treur, 1995; Burmeister & Sundermeyer, 1990; Crabtree & Jennings, 1996; Epstein & Axtell, 1996; Ferber, 1999; Gilbert & Conte, 1995). Despite a number of attempts to devise general models (Ferber & Gutknecht, 1998; Kendall, Malkoum, & Jiang, 1995; Müller, 1996), there has been no architectural standardization. Multi-agent systems draw their inspiration from Minsky (1985), who developed the idea of a *society of mind*, according to which a mind can be constructed from numerous small parts each of which is mindless. According to this principle, in a multi-agent system, competence is not centralized but distributed among different agents who communicate with each other. The key notions are usually collaboration, competition, communication, and self-organization. These models have been applied, for instance, to modelling economic and social problems (Axelrod, 1997), and articles devoted to them are abundant today in journals of computer science theory.

In the model proposed here, we assume that each dimension of a concept is identified by a single agent, and that information must be exchanged until the example presented can be identified as a positive or negative example.

The general problem is one of going from distributed knowledge to common knowledge (Fagin, Halpern, Moses, & Vardi, 1995). Based on the choice of a few basic properties related to how working memory functions, we shall present three versions of the model: parallel, serial, and random.

Let us start from the following assumptions:

1. Each agent has information about a binary dimension (so there are as many agents as there are dimensions) and knows the composition of the positive and negative subsets of examples. If, for instance, a *small red circle* is presented, the size agent knows *small* (not *big*), the color agent knows *red* (not *blue*), and the shape agent knows *circle* (not *square*). But each agent is unaware of what the others know, which means that full knowledge of the concept is distributed among them.

2. Agents take turns making the information they have public. The process ends when the publicly shared information suffices for someone who knows the composition of the positive and negative subsets (i.e., the concept) – and this is the case for all agents – to assign the concerned exemplar to the right subclass.

3. As common knowledge is being built, speaking turns are assigned on the basis of an entropy calculation, which enables agents to compare the amounts of information they are capable of contributing (see Quinlan, 1986, for a method of calculating entropy suited to the construction of trees that minimize the information flow). If a fixed rank for communicating information is set for each agent, identifying the example would amount to finding the path in a decision tree called an OBDD or ordered binary decision diagram if the dimensions are Boolean (see Bryant, 1986; Ruth & Ryan, 2000). This option will be chosen below when we develop the serial multi-agent model. The originality of the parallel multi-agent model (which will be described to the greatest extent in this study) lies in the fact that speaking turns taken by agents to release their partial knowledge are not controlled a priori but are calculated at each occurrence. When an agent releases a piece of information, it knows how much its contribution reduces the uncertainty, since it knows the target subclasses of positive and negative examples of the concept and it also knows what subset (and the positive and negative examples it contains) it is leaving for the agents that follow it. Take a three-dimensional space whose dimensions are shape (hereafter labeled F for form, C for color and S for size). If the subclass of positive examples includes all *small round* examples, *small red* examples, and *red square* examples (see Figure 1), and a *small round blue* example is presented, the shape agent, by stating "*round*" cuts the uncertainty in half (two pieces of information suffice to identify the example and it gave one; the other will be size). But the color agent will only reduce the uncertainty by a third since the other two will have to express themselves after he does. In this example, the shape agent or the size agent can thus declare a greater reduction in uncertainty than the third agent can, and one of them (say, drawn at random) will speak first. Turn-taking is thus determined by the information furnished and not by the type of dimension evoked. Thus, for a given concept, different examples can give rise to different speaking turns. If, for the concept described above, the *small red square* example is presented, the shape agent will still speak first, but this

time, it is the color agent and not the size agent who speaks in second place. The identification of examples in the case of pre-determined speaking turns would be written as follows:

$$F \wedge (S \vee C)$$

Now writing this in terms of speaking turns with maximal informativeness, we get

$$X \wedge Y$$

where X, Y, etc. are the agents (of any kind) who speak first, second, etc. In other words, no matter what example is presented, two pieces of information will suffice to assign it to the correct subclass. The speaking order of the agents thus depends upon the amount of information contributed: the most informative speaks first (or, in case of a tie, one of the most informative), and so on. To make this process more concrete, imagine a card game where, before a round, each agent states how much he will reduce the uncertainty by making a bid or laying down a card. We call this model parallel, not because the agents talk at the same time but because they simultaneously and independently calculate how much they can each reduce the entropy.

4. During knowledge building, silences are not interpreted in example assignment. Suppose the positive subclass is *big square* figures, *big blue* figures, and *blue square* figures. The example presented is a *big red round* figure. By announcing "*big*", the size agent makes it impossible for the other two agents to individually give a decisive piece of information. If they mutually interpret their silence (as in the "Game of Hats"; see Nozaki & Anno, 1991), they would arrive at the conclusion that it is a *red round* figure. Another important feature of the model is the mandatory dissociation between the information an agent thinks it will contribute and the information it actually does contribute. Sometimes the two coincide and sometimes uncertainty remains (as in the above example). Before systematically developing this point, let us take another example from the concept in Figure 1. If a *small red round* example is presented, the shape agent knows that one speaking turn will suffice after his own. If, for that example, the color agent had spoken first, then it would be uncertain about its real contribution: when it sees *red*, it can say to itself (remember that each agent knows the target subclasses) that if the example is *small* or *square*, a single speaking turn will suffice, but if it is *big* and *round*, then two speaking turns will be necessary. It cannot in fact know exactly how many agents will have to speak about a particular example after it does; it can only know this in terms of an expectancy over a large number of speaking turns.

Appendix 2. Removing Uncertainty

With one dimension, the uncertainty is completely and immediately removed because the agent in control of the dimension possesses 100% of the information. With two dimensions, there are several possible cases whose distribution is indicated in Figure 8. In case C, a single agent, hereafter

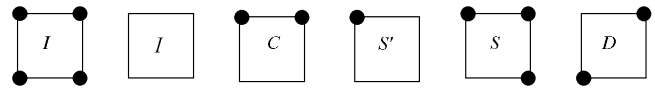


Figure 8. Uncertainty cases with 2D concepts. *Note.* Identification achieved (no speakers necessary). C: Choice (one speaker suffices). S and S': simple interaction (one or two speakers necessary, depending on the case). D: dual interaction (two speakers always necessary).

denoted X, suffices to remove the indetermination. (Remember that labels are assigned to speaking turns, not particular agents: X is not the color agent or the shape agent but the agent who speaks first, because it is the one (or among the ones) who provides the most information.) In case D, whether a positive or negative example is at stake, two agents must speak up: X and Y. In case S, identifying one of the three positive examples requires only one agent because the three examples are in a disjunctive relation (e.g. *square* or *red*). On the other hand, identifying the negative example requires the participation of two agents (as in the above *blue round* example). In case S, we therefore sometimes have X (3/4) and sometimes $X \wedge Y$ (1/4). This is a particular disjunction since it is $X \vee Y$ with X always present. Using the notation of propositional logic, this case is written $X\ [\ Y\ ]$ (affirmative connective of X, for all Y) whose truth table is:

X	X[Y]	Y
1	1	1
1	1	0
0	0	1
0	0	0

Case S' has exactly the same structure as case S, except for the fact that the proportions of positive and negative examples are reversed. Case I is trivial - no information is necessary. Case D is a complete interaction because the decision cannot be made, regardless of what one agent responds, unless he knows the other agent's response. Cases S and S' are partial interactions between the two dimensions. Thus, when just two dimensions are considered, there are only three ultimate forms of inter-agent communication. They are written:

$$X;\ X[Y];\ X \wedge Y$$

Whenever additional dimensions are considered, they will be either necessary or optional. One can thus deduce that in a space with n binary dimensions, an inter-agent communication process that progresses from implicit distributed knowledge to enough common knowledge to assign an example to the correct subclass is an expression that only supports the operators $\wedge$ (necessary) and [ ] (optional).

The sixteen binary operators of propositional logic can be related (with one rotation) to two-dimensional concepts, as indicated in Figure 9.

Piaget's INRC group, or the Klein four-group is a group of 4 elements that combines two cyclical, 2-element groups.

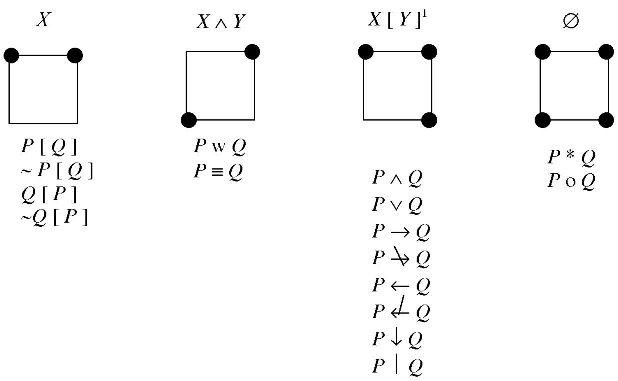


Figure 9. Concepts and propositions.

Table 4
Conceptual forms up to four dimensions.

D1	D2	D3	D4
X	$X[Y]$	$X[Y[Z]]$	$X[Y[Z[W]]]$
			$X[Y[Z \wedge W]]$
		$X[Y \wedge Z]$	$X[Y \wedge Z[W]]$
			$X[Y \wedge Z \wedge W]$
	$X \wedge Y$	$X \wedge Y[Z]$	$X \wedge Y[Z[W]]$
			$X \wedge Y[Z \wedge W]$
		$X \wedge Y \wedge Z$	$X \wedge Y \wedge Z[W]$
			$X \wedge Y \wedge Z \wedge W$

If we add a third 2-element group, we obtain an 8-element group, called the extended INRC group (Apostel, 1963). The extended group operates within the sixteen operators of propositional logic, with the same categories as in Figure 9. One can understand this for geometrical reasons (Klein four-group operators are simple or combined rotations in this case).

The relevance of the notation in terms of necessary and optional dimensions can easily be verified. For instance, for all junctions and all implications, it may suffice to have a single piece of information (e.g., p false for $p \wedge q$; p true for $p \vee q$; q true for $p \downarrow q$; q false for $p \leftarrow q$, etc.), but it is necessary to have two pieces in half of the cases (this can easily be seen by inverting the truth values in the above examples in parentheses).

Starting from these three basic expressions in two dimensions, each formula is obtained by adding a new agent, connected by one of the two operators. This is a recursive concept-building process that justifies the name "graceful" complexification. Table 4 gives the different forms up to four dimensions.

The formula of a concept condenses the set of operations applicable to each positive and negative example of the concept. It is easy to find them and list them in disjunctive form. For instance, for $X[Y \wedge Z[W]]$, we get:

$$X \vee (X \wedge Y \wedge Z) \vee (X \wedge Y \wedge Z \wedge W)$$

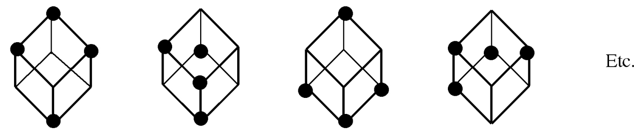


Figure 10. Rotations and enantiomorphisms.

In other words, each example requires the contribution of either a single agent, or of three or four agents. Another example is:

$$X[Y[Z[W]]] \equiv X \vee (X \wedge Y) \vee (X \wedge Y \wedge Z) \vee (X \wedge Y \wedge Z \wedge W)$$

Appendix 3. Counting and Characterizing Concepts

Before coming back to how our multi-agent system works, let us make a few remarks about concepts with up to three and sometimes four dimensions. Note that the fourth dimension is generally the upper limit of human working memory, not because the boundary between 3 and 4 or between 4 and 5 is decisive, but because one must consider the entire set of relations between the elements in working memory. So it is not a load E of elements that has to be considered but a load of $P(E)$, that is, all subsets of n elements, since every subset also constitutes an example of the concept. Three elements generate a load of 8, 4 a load of 16, 5 a load of 32, etc. This approach has now been adopted by authors desirous of reconciling Piaget's theory and information-processing theory (the so-called neo-Piagetian trend) and who, following Pascual-Leone's (1970) seminal work, developed various models revolving around the new "magical number" four (Case, 1985; Cowan, 2001, though not neo-Piagetian; Fischer, 1980; Halford, Wilson, & Phillips, 1998). We find this natural limitation in many human activities, such as the four main phrases of a sentence, the four singing voices, the four suits in a deck of cards.

Every concept possesses numerous realizations in its space that are equivalent save one transformation (rotations and enantiomorphisms, Fig. 10) and one substitution of the subclasses of positive and negative examples (e.g. the set of *small round* and *red square* examples is equivalent to the set of *big round* and *blue square* examples insofar as the same subclasses are opposed). For instance:

Thus, many concepts are equivalent. Table 5 gives a count of concepts up to four dimensions.

Figure 11 gives a count of the equivalent forms for three dimensions. Each concept is labelled with an identification number (in boldface), which will be used in the remainder of this article.

Table 6 gives a count of the different forms in three dimensions. It does not bring out any notable regularities.

Appendix 4. Multi-Agent System and Terminating the Identification Process

To describe the functioning of the multi-agent system in greater detail, we devised a table that explains the speaking

Table 5
Number of different concepts up to four dimensions.

	Number of positive examples														
Dimensions	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1														
2	1	2	1												
3	1	3	3	6	3	3	1								
4	1	4	6	19	27	50	56	74	56	50	27	19	6	4	1

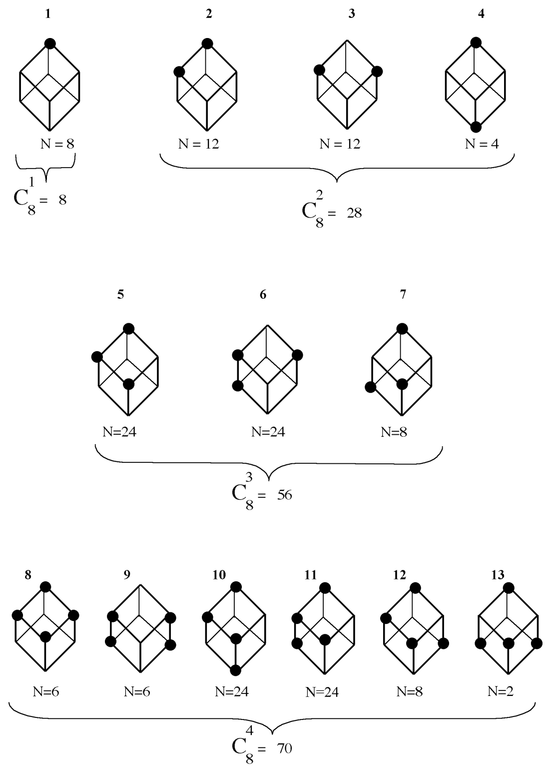


Figure 11. Equivalent forms in three dimensions.

Table 6
The 13 concepts in three dimensions.

	Number of positive examples			
Formula	1	2	3	4
X				*
$X[Y]$		*		
$X \wedge Y$				**
$X[Y[Z]]$			*	
$X \wedge Y[Z]$		*	**	**
$X[Y \wedge Z]$	*	*		
$X \wedge Y \wedge Z$				*

turns, and we present some illustrations for the parallel version. Table 7 shows Concept 12. The rows contain the examples of the vertices of a cube, numbered as in Figure 1B, which will be used here for all cases presented (the positive examples of Concept 12 are 1, 5, and 6).

The first three columns give the situation the agent in that column leaves for the other two agents if he speaks about the example in that row. For three dimensions, the first speaker leaves a *square* which can only have four forms, as we saw above: identification achieved (0 speakers) denoted *I* (though it is not used in the present tables), dual interaction (2 speakers) denoted *D*, simple interaction (1 or 2 speakers) denoted *S*, and choice (1 speaker) denoted *C*. The ambiguous case is the simple interaction, where the agent who talks does not knows exactly what reduction of uncertainty he brings (but only knows it in a probabilistic manner). For each example in the table, after the simple interaction (*S*), the real situation is shown in brackets (*D* or *C*). It is normal that an agent who leaves a simple interaction expresses itself before an agent leaving a dual one, although his actual uncertainty reduction will not be greater in one out of four cases. The Table 7 allows us to write the identification formula: $X \wedge Y[Z]^4$. This formula means that the contribution of two agents is always necessary, and that it is sufficient only for four of the eight examples. Four times then (index of *Z*), the third agent will have to speak. The same principles apply to Concept 11 (Table 8).

Let us look in detail at a final case, Concept 10 (Table 9). The situation is very revealing about the uncertainty effects generated by a simple interaction. In all cases, it would be possible to identify the example with only two agents. Yet it is impossible for the system to know this at the onset, and all agents are in the same situation, that of leaving a simple interaction, with 2/3 leading to a choice and 1/3 leading to a dual interaction on examples 1, 3, 4, 5, 6, and 8. All in all, since 6 x 1/3 = 2, the formula is $X \wedge Y[Z]^2$. Although this case is different from the preceding ones, themselves different from each other, their formulas are identical (disregarding the index in certain cases). We will say that they are isotopes for the purposes of our classification.

Concept 10 is exemplary of the ambiguities that arise when we go from a state of distributed knowledge to a state of common knowledge in a community of agents. To better illustrate this ambiguity, let us borrow an example from Fagin, Halpern, Moses, and Vardi (1995), who gave the example of three wives likely to be cheated on by their husbands but each one having knowledge only of the misfortune of the others, should the case arise. The state of the world is $A+\ B-$

Table 7
Communication table for the examples of Concept 12 (ex+ = 1, 5, 6).

	Size (S)	*Color* (C)	*Shape* (F)	Speaks 1st	Speaks 2nd	Speaks 3rd
1	D	$S[D]$	D	C	$S \vee F$	$F \vee S$
2	D	D	D	$S \vee C \vee F$	$S \vee C \vee F$	$S \vee C \vee F$
3	$S[C]$	$S[C]$	D	$S \vee C$	$S \vee C$	****
4	D	$S[C]$	$S[C]$	$C \vee F$	$C \vee F$	****
5	$S[D]$	D	D	S	$C \vee F$	$F \vee C$
6	D	D	$S[D]$	F	$S \vee C$	$C \vee S$
7	$S[C]$	$S[C]$	$S[C]$	$S \vee C \vee F$	$S \vee C \vee F$	****
8	$S[C]$	D	$S[C]$	$S \vee F$	$F \vee S$	****

Note. This table gives the situation after the first speaking turn. A similar table is needed for the second turn; the reader can easily guess what it would be.

Table 8
Communication table for Concept 11 (ex+ = 1, 2, 6, 7).

	Size (S)	*Color* (C)	*Shape* (F)	Speaks 1st	Speaks 2nd	Speaks 3rd
1	$S[C]$	$S[D]$	C	F	S	****
2	$S[C]$	C	C	$C \vee F$	S	****
3	$S[C]$	D	C	F	S	****
4	$S[D]$	$S[D]$	$S[D]$	$S \vee C \vee F$	$S \vee C \vee F$	$S \vee C \vee F$
5	$S[C]$	C	C	$C \vee F$	$F \vee C$	****
6	$S[C]$	C	D	C	S	****
7	$S[D]$	D	D	S	$C \vee F$	$F \vee C$
8	$S[C]$	C	D	C	S	****

Note. This table gives the situation after the first speaking turn. A similar table is needed for the second turn; the reader can easily guess what it would be.

$C+$ (i.e., A and C have unfaithful husbands). In this system, we can describe the following two levels for knowledge Kn = "There is at least one wife whose husband is unfaithful":

1. *Mutual knowledge*:
$\forall i, j, k \in E = (A, B, C)$, i knows Kn, and knows that j knows Kn and that j knows that k knows Kn.

2. *Shared knowledge*:
$\forall i \in E = (A, B, C)$, i knows Kn. Each agent knows that there is at least one $+$ but doesn't know that the others know. The present example stabilizes in this state if no information is communicated. Individually, by way of her perception of the situation, each of the three wives considers the proposition "There is at least one $+$" to be true. But she cannot infer anything about what the others know. A could very well think that B only sees one $+$ (if A thinks she is $-$ herself), that C sees no $+$'s and that she believes there are none (if A thinks she is $-$ herself and assumes that C does likewise). We end up with a considerable distortion between the state of the world ($A+$ $B-$ $C+$) and the attribution of A's belief to C ($A-$ $B-$ $C-$).

Appendix 5. Galois Lattices

A complexity hierarchy for concepts up to three dimensions can be proposed based on the ordering of multi-agent formulas in a Galois lattice (Figure 12).

A lattice is an ordered set in which any two elements always have an upper bound and a lower bound (Davey & Priestley, 1990). Let a and b be two elements, the upper bound ($a \cup b$) is the supremum of these two elements. Reasoning in the same way for the lower bound, the infemum of a and b is ($a \cap b$). More specifically in the framework of formal conceptual analysis, two lattices are merged to form pairs and this gives a Galois lattice (Ganter & Wille, 1991). Each multi-agent formula is seen as a pair (A, B) such that A is the set of concepts learnable from a communication formula (definition in extension) and B is the set of constraints imposed upon a formula (definition in intension). In the lattice, each formula assigned to a location can learn all subordinate concepts. For simplicity's sake, across from each formula, we give only the most complex concept learnable from it. The processing cost is incurred when equivalent communication structures (isotopes) process different concepts (e.g. Concepts 8, 9, 10, 11, and 12 for the structure $X \wedge Y[Z]^n$). In this case, the structure occurs several times with different cost indexes. The criterion that orders the structures is:

$$(A1, B1) < (A2, B2) \equiv A1 \subseteq A2 \equiv B2 \subseteq B1$$

Thus, a complex communication protocol enables learning of more concepts than a protocol that is subordinate to it. The more constraints one imposes on a communication protocol (fewer agents, less communication), the smaller the number of concepts the protocol can learn. Inversely, the

Table 9
Communication table for Concept 10 (ex+ = 1, 2, 5, 6).

	Size (S)	*Color* (C)	*Shape* (F)	Speaks 1st	Speaks 2nd	Speaks 3rd
1	$S[C]$	$S[D]$	$S[C]$	$S \vee C \vee F$	$S \vee C \vee F$	$(S \vee F)$
2	$S[C]$	$S[C]$	$S[C]$	$S \vee C \vee F$	$S \vee C \vee F$	****
3	$S[C]$	$S[C]$	$S[D]$	$S \vee C \vee F$	$S \vee C \vee F$	$(S \vee C)$
4	$S[D]$	$S[C]$	$S[C]$	$S \vee C \vee F$	$S \vee C \vee F$	$(C \vee F)$
5	$S[D]$	$S[C]$	$S[C]$	$S \vee C \vee F$	$S \vee C \vee F$	$(C \vee F)$
6	$S[C]$	$S[C]$	$S[D]$	$S \vee C \vee F$	$S \vee C \vee F$	$(S \vee C)$
7	$S[C]$	$S[C]$	$S[C]$	$S \vee C \vee F$	$S \vee C \vee F$	****
8	$S[C]$	$S[D]$	$S[C]$	$S \vee C \vee F$	$S \vee C \vee F$	$(S \vee F)$

Note. This table gives the situation after the first speaking turn. A similar table is needed for the second turn; the reader can easily guess what it would be.

fewer the constraints, the more a formula is able to learn numerous, complex concepts (even if the available agents or communications become useless for simpler concepts). Starting from the top of the lattice, we find a concept (A, B) such that B does not contain the constraint present in the concept below it (e.g., compared to the formula $X \wedge Y \wedge Z$, we impose only 4 calls on Z for the concept $X \wedge Y[Z]^4$). This process is repeated in a top-down manner until all constraints have been used (we then obtain a unary agential protocol: X). The concepts are treated in the same way via a bottom-up process: moving up the lattice, we gradually add the learnable concepts (this means that the formula $X \wedge Y \wedge Z$ permits learning of all concepts in three dimensions). This lattice provides a theoretical concept-complexity order. An analogy can be drawn with the complexities described in the introduction and used in computer science theory. The total number of agents in a formula corresponds to the maximal compression of the group of speakers. This number is like random Chaitin-Kolmogorov complexity; it is the cardinal of the set of all necessary independent agents. Then, during the identification of all examples of the concept, the agents will be called a certain number of times. This reuse of agents can be likened to Bennett's logical depth[4].

[4] Remember that Chaitin-Kolmogorov complexity corresponds to the size of the smallest program capable of describing (computing) an object. This size can be measured in terms of information quantity. Now, Bennett's logical depth corresponds to the computation time of this smallest program. In other words, it is proportional to the number of times the different routines will be called. A Tower

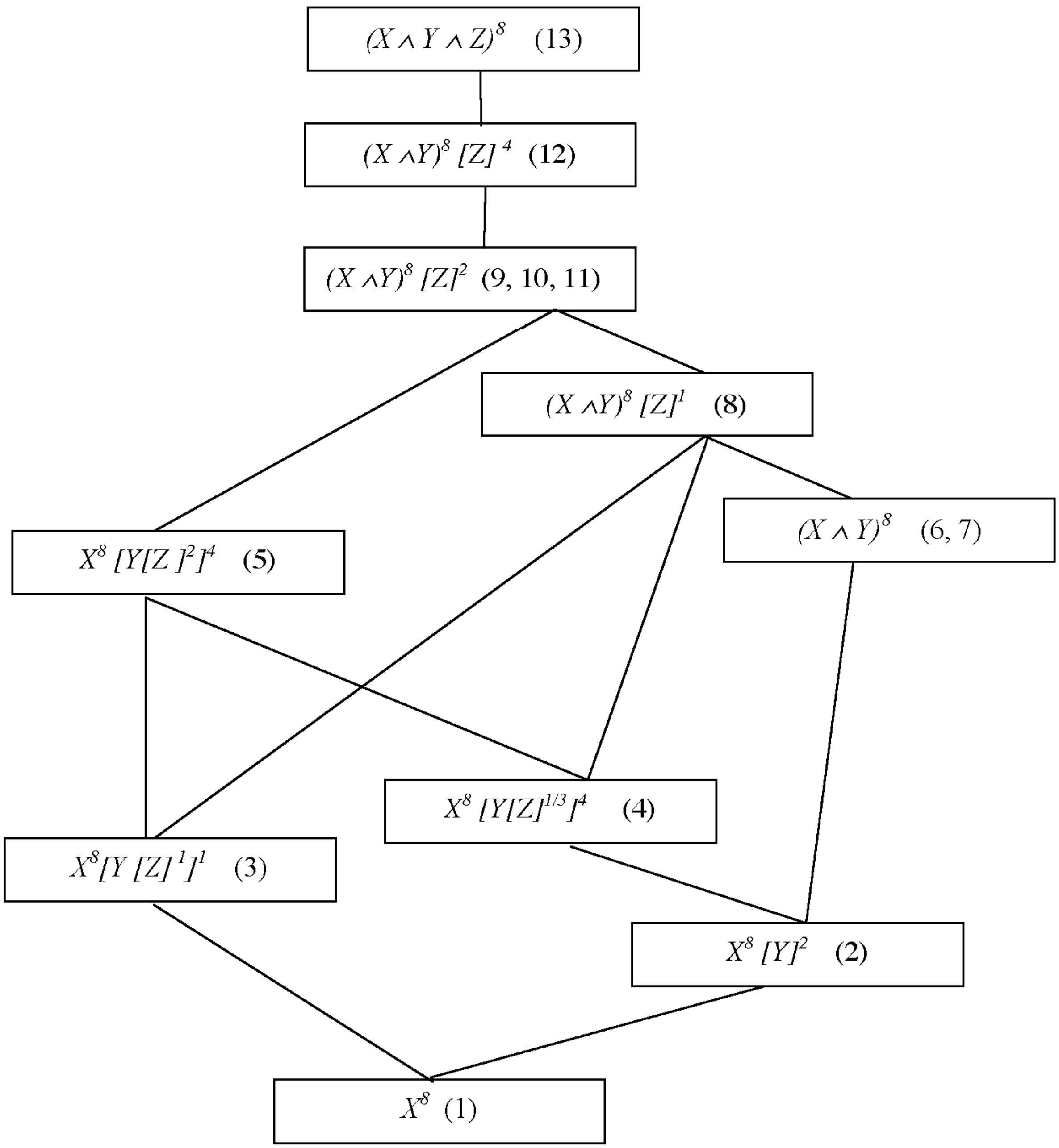


Figure 12. Galois lattice of concepts in the parallel version.